Chapter (2)

Conduction

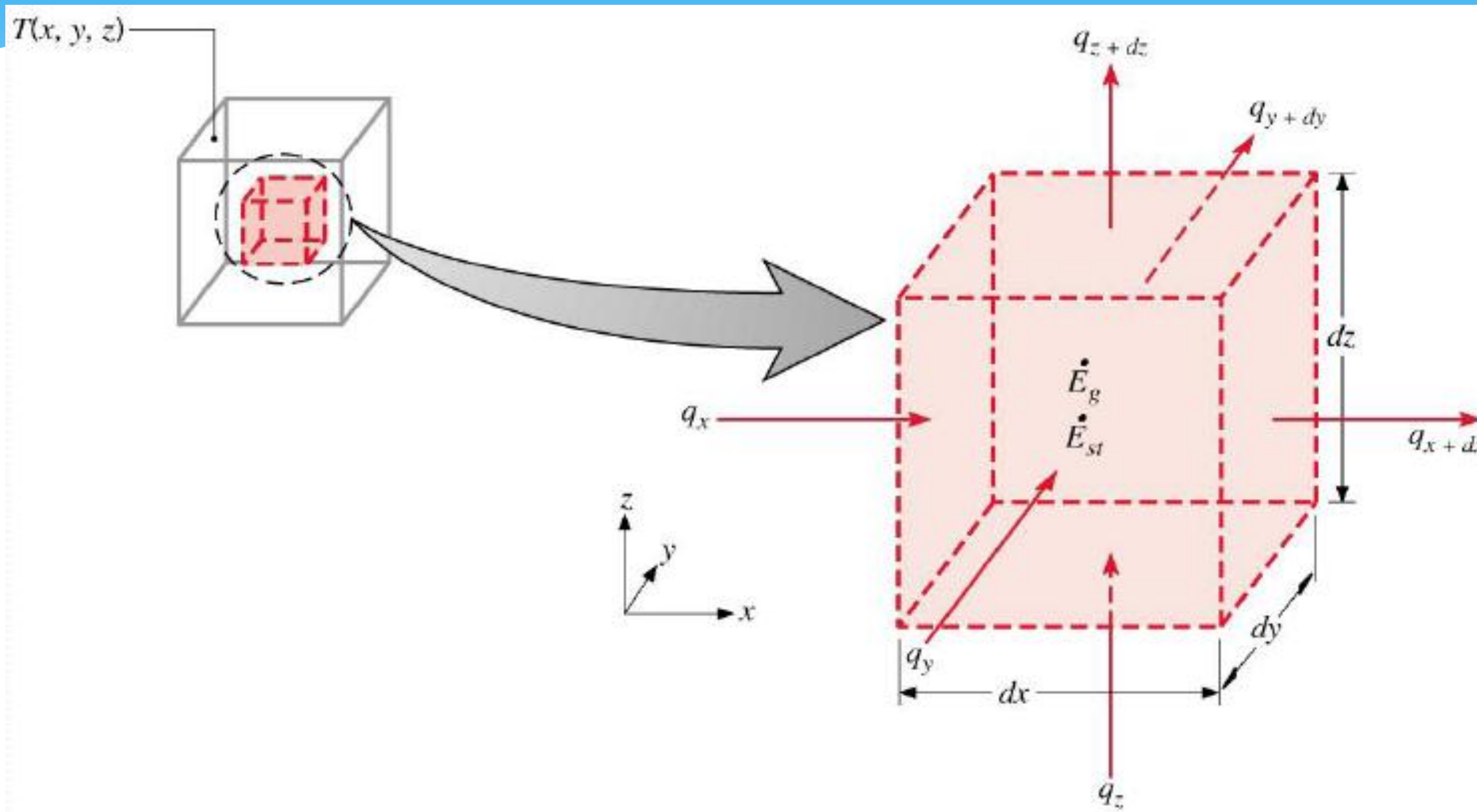
Goals:

By the end of today's lecture, you should be able to:

- Learn how to obtain temperature profiles for common geometries without heat generation.
- Learn how to obtain heat flow for different applications under the assumptions of steady state and one-dimensions
- You will be familiar with the concept of thermal resistance, thermal circuits and overall heat transfer coefficients.

The Heat Diffusion Equation (General Form)

Cartesian Coordinates T(x,y,z)



We will apply the energy conservation equation to the differential control volume

Cartesian Coordinates T(x,y,z)

Energy in = Energy out

$$\sum q_{in} + \dot{E}_{gen} = \sum q_{out} + \dot{E}_{st}$$

Where:

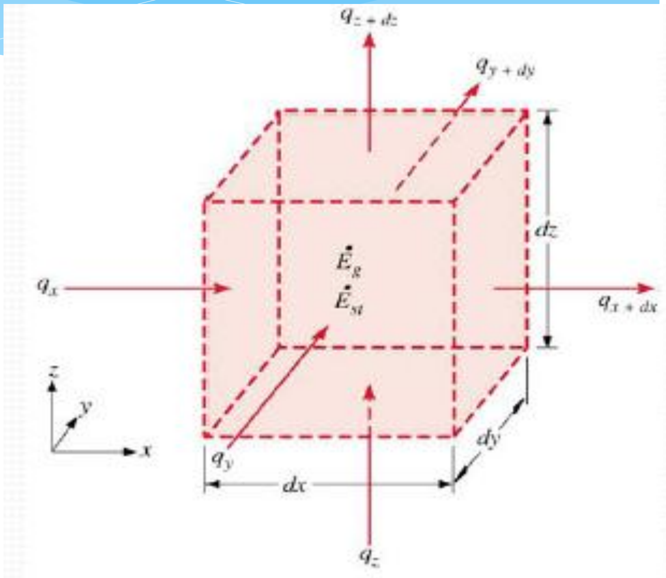
$$\sum q_{in} = q_x + q_y + q_z$$

From Fourier's law, the conduction heat transfer rates perpendicular to each of the control surfaces at the x, y, z coordinate locations are:

$$q_x = -kA_x \frac{\partial T}{\partial x} = -k(dydz)\frac{\partial T}{\partial x}$$

$$q_y = -kA_y \frac{\partial T}{\partial y} = -k(dxdz)\frac{\partial T}{\partial y}$$

$$q_z = -kA_z \frac{\partial T}{\partial z} = -k(dxdy)\frac{\partial T}{\partial z}$$



Cartesian Coordinates T(x,y,z)

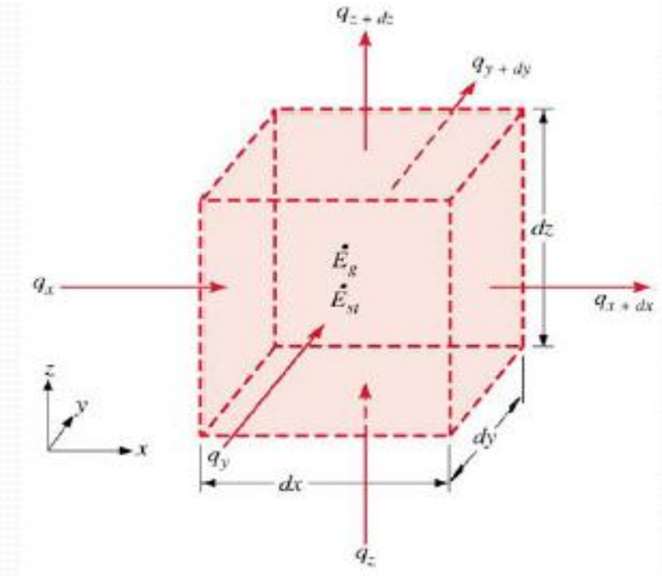
$$\sum q_{out} = q_{x+dx} + q_{y+dy} + q_{z+dz}$$

From Taylor series expansion where, neglecting higher order terms, the conduction heat rates at the opposite surfaces can be expressed as:

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx$$

$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz$$



Cartesian Coordinates T(x,y,z)

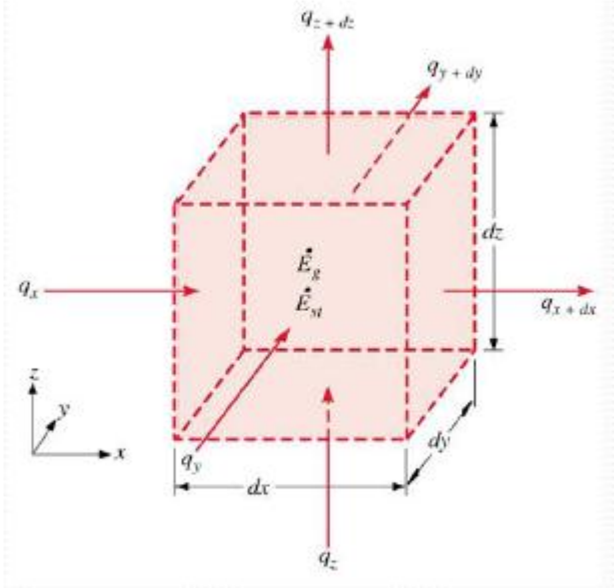
Thermal energy generation due to an energy source:

- Positive (source) if thermal energy is generated
- Negative (sink) if thermal energy is consumed

$$\dot{E}_g = \dot{q}\, \mathrm{dV} = \dot{q}(dx\, dy\, dz)$$

Where:

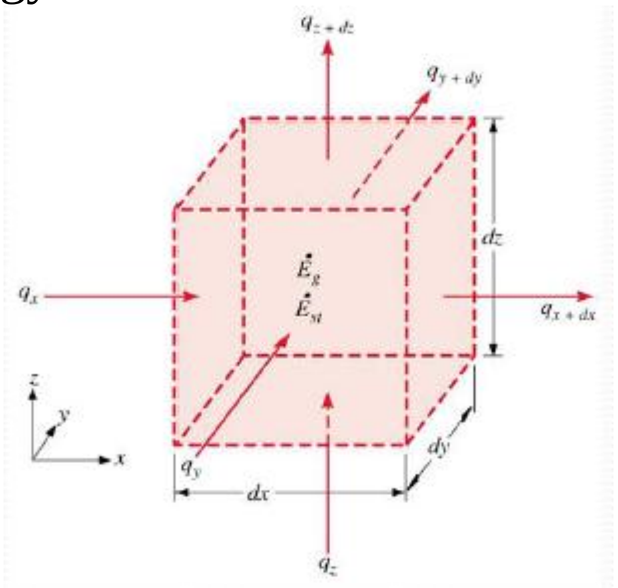
$\dot{q}$ is the rate at which energy is generated per unit volume of the medium (W/m^3)



Cartesian Coordinates T(x,y,z)

Energy storage term

Represents the rate of change of thermal energy stored in the matter in the absence of phase change.



$$\dot{E}_{st} = m\,c_p \frac{\partial T}{\partial t} = \rho V c_p \frac{\partial T}{\partial t}$$

$$\dot{E}_{st} = \left[\rho c_p \frac{\partial T}{\partial t}\right](dx\,dy\,dz)$$

$\rho c_p \dfrac{\partial T}{\partial t}$ is the time rate of change of the sensible (thermal) energy of the medium per unit volume (W/m^3)

Cartesian Coordinates T(x,y,z)

Substituting all terms into Eq.

$$\underbrace{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)}_{\text{Net transfer of thermal energy into the control volume (inflow-outflow)}}+\underbrace{\dot{q}}_{\text{Thermal energy generation}}=\underbrace{\rho c_p\frac{\partial T}{\partial t}}_{\text{Change in thermal energy storage}}$$

At any point in the medium the rate of energy transfer by conduction into a unit volume plus the volumetric rate of thermal energy generation must equal the rate of change of thermal energy stored within the volume

Cartesian Coordinates T(x,y,z)

- **If k = constant**

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

$$\boxed{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

$$\alpha=\frac{k}{\rho c_p}$$ is the thermal diffusivity

Thermal diffusivity : is the ratio of the thermal conductivity to the heat capacity.

Cartesian Coordinates T(x,y,z)

- **For steady state conditions**

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\cancel{\frac{\partial T}{\partial t}}$$

$$\boxed{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=0}$$

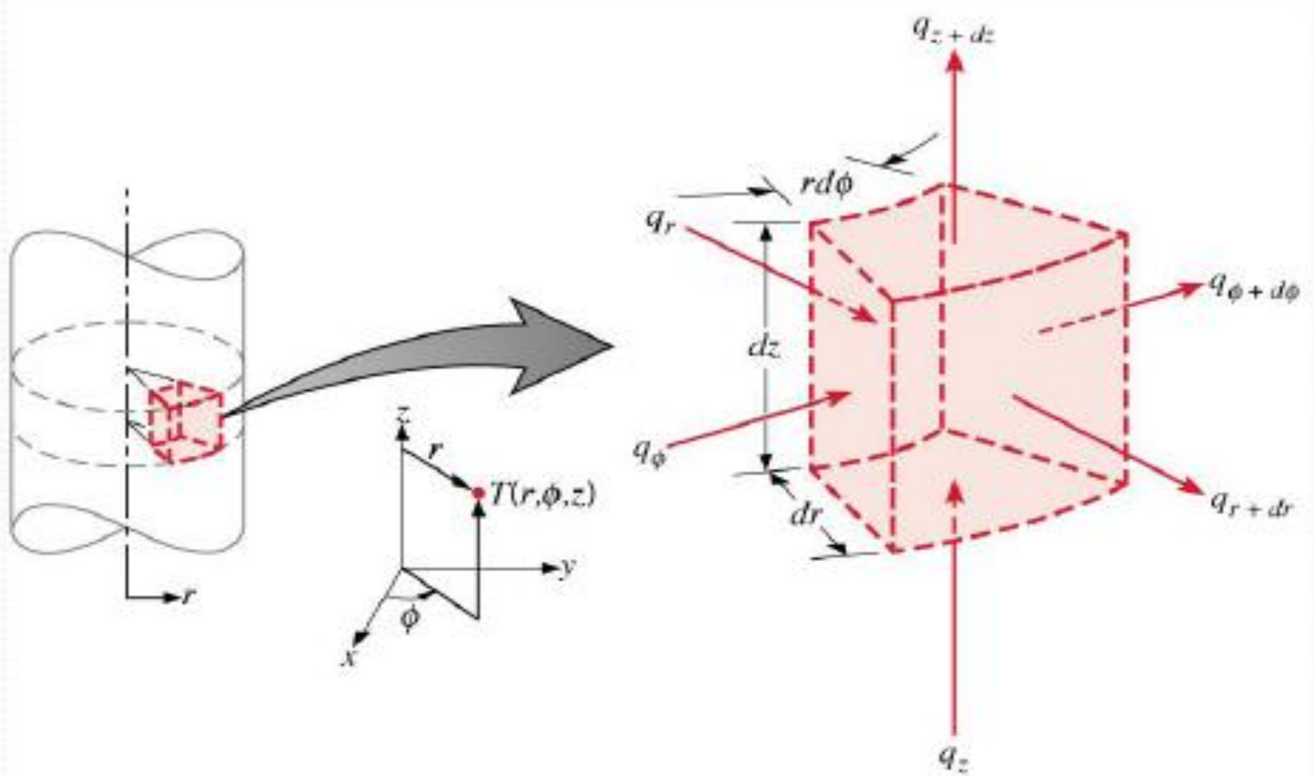
Cartesian Coordinates T(x,y,z)

- **For steady state conditions, one-dimensional transfer in x-direction and no energy generation**

$$\boxed{\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0 \quad \text{or} \quad \frac{dq''_x}{dx}=0}$$

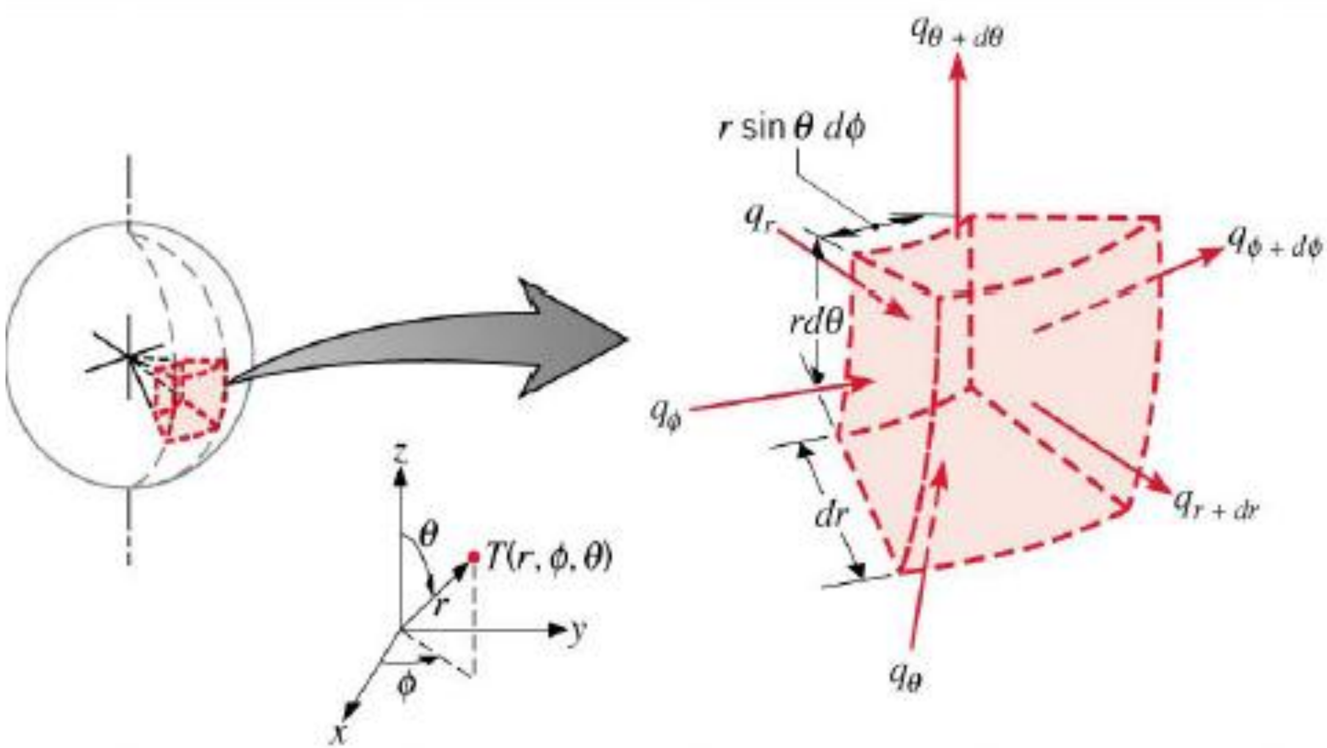
Heat flux is constant in the direction of transfer

Cylindrical Coordinates T(r,ϕ,z)



$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

Spherical Coordinates T(r,ϕ,Θ)



$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

Applications

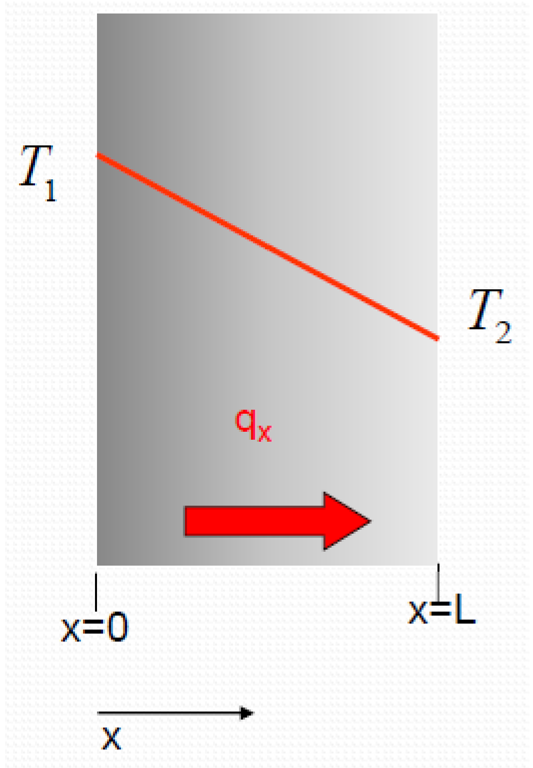
One Dimensional
Steady State Conduction

Plan Wall

Heat transfer across a rectangular solid is the most direct application of Fourier's law.

Consider a simple case of one-dimensional conduction in a plane wall, with the following assumptions:

- No heat sources
- Constant thermal conductivity
- One dimension (x- direction)
- Steady state
- Cross-sectional area is not dependent on the x-coordinate
- Thermal boundary layers on each face (x = 0 and x = 1)



Plan Wall

From the general equation:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

Apply the previews assumptions

$$\frac{\partial^2 T}{\partial x^2}=0 \longrightarrow \frac{d^2 T}{dx^2}=0 \qquad \text{Where } T = T(x)$$

Solution of the differential equation

$$\frac{dT}{dx}=C_1 \longrightarrow T(x)=C_1\,x+C_2$$

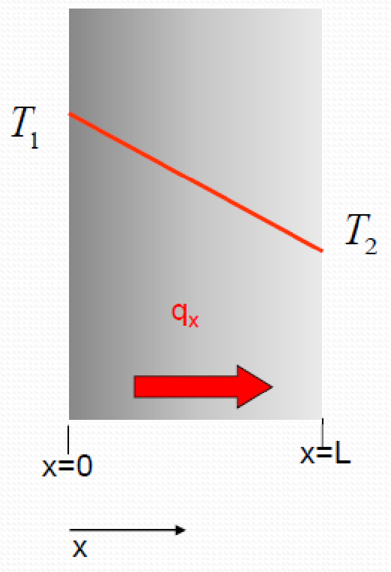
Plan Wall

Boundary conditions. . .

At $X = 0$ $\longrightarrow$ $T = T_1$ $\longrightarrow$ $T_1 = C_2$ T_1

At $X = L$ $\longrightarrow$ $T = T_2$ $\longrightarrow$ $T_2 = C_1 L + T_1$

$\longrightarrow$ $C_1 = (T_2 - T_1)/L$



Temperature distribution. . .

$$\boxed{T(x) = -\left(\frac{T_1 - T_2}{L}\right)x + T_1}$$

From Fourier's law. . .

$$\boxed{q_x = -kA\frac{dT}{dx}}$$

Plan Wall

Substitute by dT/dx in Fourier's law. .

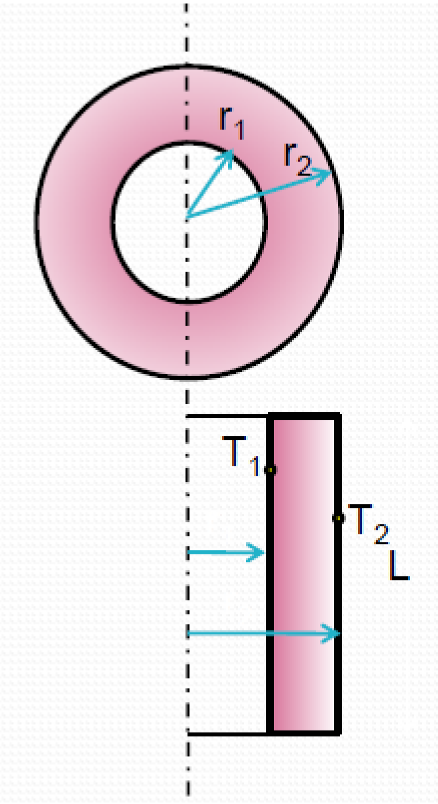
$$q_x = -kA\frac{dT}{dx} = -kA\left(-\frac{T_1 - T_2}{L}\right) = kA\left(\frac{T_1 - T_2}{L}\right)$$

$$\therefore \; q_x = \frac{T_1 - T_2}{\left(\dfrac{L}{kA}\right)}$$

Radial Systems

Consider a simple case of one-dimensional conduction in a pipe of finite thickness, with the following assumptions:

- No heat sources
- Constant thermal conductivity
- One dimension (x-direction)
- Steady state
- Cross-sectional area is not dependent on the x-coordinate
- Thermal boundary layers (r = r1and r = r2)



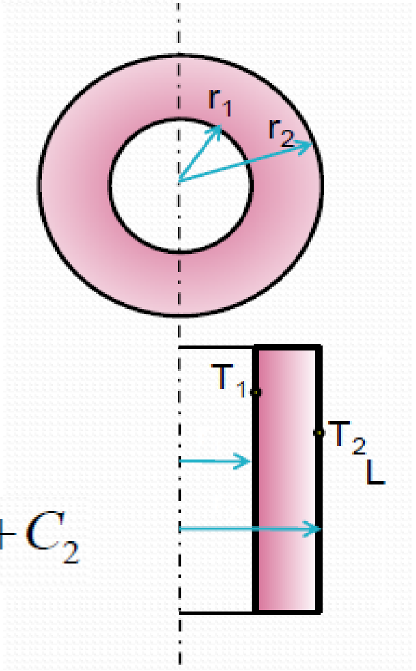
Radial Systems

From the general equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

Apply the previews assumptions

$$\therefore \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$$



Solution of the differential equation

$$\therefore \quad r\frac{dT}{dr}=C_1 \longrightarrow \frac{dT}{dr}=\frac{C_1}{r} \longrightarrow \therefore \quad T(r)=C_1 \ln r+C_2$$

Radial Systems

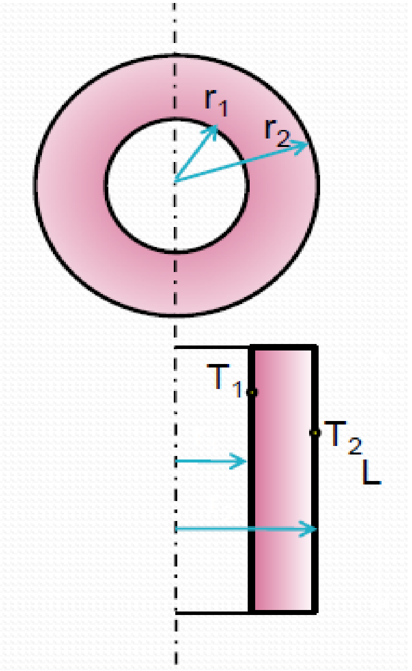
Boundary conditions. . .

At $r = r_1$ $\longrightarrow$ $T = T_1$ $\longrightarrow$ $T_1 = C_1 \ln r_1 + C_2$

At $r = r_2$ $\longrightarrow$ $T = T_2$ $\longrightarrow$ $T_2 = C_1 \ln r_2 + C_2$

Temperature distribution. . .

$$\boxed{T(r) = -\left(\frac{T_1 - T_2}{\ln \frac{r_2}{r_1}}\right) \ln r + T_1 + \left(\frac{T_1 - T_2}{\ln \frac{r_2}{r_1}}\right) \ln r_1}$$

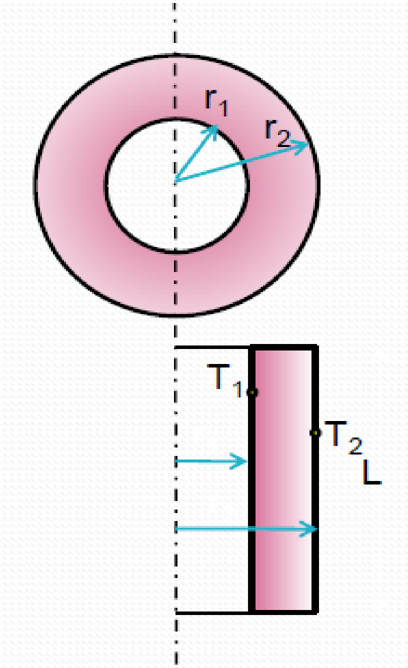


Radial Systems

From Fourier's law. . .

$$q_r = -kA\frac{dT}{dr} = -k\,(2\pi rL)\left(-\frac{T_1 - T_2}{\left(\ln\frac{r_2}{r_1}\right)r}\right)$$

$$\Rightarrow \quad \boxed{q_r = \frac{T_1 - T_2}{\left[\dfrac{\ln\left(\dfrac{r_2}{r_1}\right)}{2\pi kL}\right]}}$$

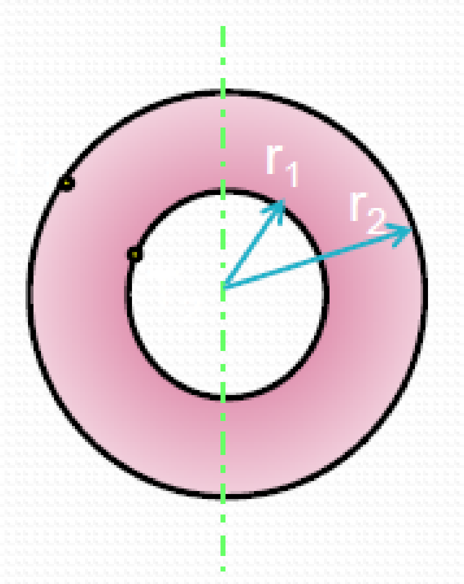


Spherical Systems

By the same method…

Prove that the heat flow in a spherical shell can be expressed by the following equation:

$$\boxed{q_r = \frac{T_1 - T_2}{\left[\dfrac{\dfrac{1}{r_1} - \dfrac{1}{r_2}}{4\pi k}\right]}}$$



Equivalent Resistance Method

Many simple heat transfer problems can be solved using the "**equivalent thermal circuit**" **method.**

It is possible to compare heat transfer to current flow in electrical circuits.
Where:

The **heat transfer rate** may be considered as a **current flow** and the combination of **thermal conductivity**, **thickness of material**, **and area** as a **resistance** to this flow. The **temperature difference** is the **potential** or driving function for the heat flow, resulting in the Fourier equation being written in a form similar to Ohm's Law of Electrical Circuit Theory.

$$\text{Electric current} = \frac{\text{Potential Difference}}{\text{Resistance}} \longrightarrow I = \frac{V}{R}$$

$$q = \frac{T_1 - T_2}{\left(\frac{L}{kA}\right)}$$

Equivalent Resistance Method

Accordingly, from the previous discussion we can determine the thermal resistance for three common shapes as follows:

Slab (Wall): $$R_{th}\Big|_{plan} = \frac{L}{kA}$$

Cylinder: $$R_{th}\Big|_{Cylinder} = \frac{\ln\left(\dfrac{r_2}{r_1}\right)}{2\pi kL}$$

Sphere: $$R_{th}\Big|_{Sphere} = \frac{\dfrac{1}{r_1} - \dfrac{1}{r_2}}{4\pi k}$$

Equivalent Resistance Method

Thermal resistance can also be associated with heat transfer by convection;

$$q = hA(T_s - T_\infty) \Rightarrow \boxed{R = \frac{1}{hA}}$$

And with heat transfer by radiation;

$$q = \varepsilon\sigma A_s(T_s^4 - T_\infty^4) \Rightarrow q = h_{rad} A_s (T_s^4 - T_\infty^4)$$

$$\boxed{\therefore \quad R = \frac{1}{h_{rad} A_s}}$$

Equivalent Resistance Method

A surface exposed to the surrounding air involves convection and radiation simultaneously. In this case the convection and radiation resistances are parallel to each other.

When the surrounding temperature equals to T∞, the radiation effect can properly be accounted for by replacing h in the convection resistance relation by:

$$h_{combined} = h_{conv} + h_{rad} \; (W/m^2 \cdot K)$$

where $h_{combined}$ is the **combined heat transfer coefficient.**
By this way we can simplify the problem and avoid all complications associated with radiation.

Overall Heat Transfer Coefficient

$$q_x = UA\Delta T$$

$$R_{tot} = \sum R_t = \frac{\Delta T}{q} = \frac{1}{UA}$$

$$\therefore U = \frac{1}{R_t\, A}$$

where U is the overall heat transfer coefficient and ΔT the overall temperature difference.

Equivalent Resistance Method